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Spaces with no uncountable submetrisable subsets

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Abstract

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We give a new characterisation of submetrisability, and define a nonstrandable space as a T_3 uncountable space with no uncountable submetrisable subset. We give some equivalent definitions and show that if there is a nonstrandable space, then there is one that is either hereditarily separable or hereditarily Lindelöf. We characterise all hereditarily Lindelöf nonstrandable spaces, show that there is a GO nonstrandable space iff there is a Suslin line, give another example of a nonstrandable space (under $\diamond$) and show that certain classes of spaces can never be nonstrandable spaces.

Keywords: Submetrisable, hereditarily ccc, GO space, Suslin line.

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1. Introduction and definitions

In this paper I give a new characterisation of submetrisability, and apply it to the problem of finding spaces with no uncountable submetrisable subset. This problem was mentioned by Gruenhage [6], who showed that if the following is consistent:

> Every uncountable first countable regular space contains either an uncountable metrisable subspace or an uncountable subspace of the Sorgenfrey line, $(*)$

then certain problems on perfectly normal compacta can be solved. In particular he showed that

(1) every perfectly normal, compact, locally connected space is metrisable if $(*)$ holds, and

(2) if $X \times Y$ is compact and perfectly normal, then one of X and Y must be metrisable, if (*) and PFA both hold,
where PFA is Baumgartner's proper forcing axiom (see [3]). As (*) implies that each uncountable regular space has an uncountable submetrisable subset, it would be interesting to know whether there exist spaces with no uncountable submetrisable subset, either absolutely or under PFA, and whether there are any first countable such spaces.

The structure of the paper is as follows. We give a new characterisation of submetrisability in Section 2, then in Section 3 we define nonstrandable spaces[1] and use Section 2 to give various equivalent definitions. In Section 4 hereditarily ccc spaces are investigated and it is shown that there is a hereditarily Lindelöf nonstrandable space if and only if there is a hereditarily Lindelöf subset X of 2^{ω_1} such that, for each $\alpha \in \omega_1$, $\{x|_\alpha : x \in X\}$ is countable (Corollary 4.7). In Section 5 we give an example of a nonstrandable space (under $\diamondsuit$) and in Section 6 show that the nonstrandable GO spaces are precisely the Suslin lines with no uncountable separable subsets. Finally in Section 7 we show that "most" Čech-complete spaces and "most" spaces with G_δ-diagonal cannot be nonstrandable spaces.

Recall that a *submetrisable* space is one that has a weaker metric topology, and an *$\mathcal{SM}$-contractible* space is one that has a weaker separable metric topology (where $(X, \mathcal{M})$ is *weaker* than $(X, \mathcal{T})$ if $\mathcal{M} \subseteq \mathcal{T}$).

Any undefined notions can be found in [5] or in [7].

Conventions. All spaces are assumed to be regular Hausdorff, and all maps are assumed continuous. Functions are not necessarily continuous.

2. A characterisation of submetrisability

In [10], Reed and Zenor showed that if X is a perfectly normal subparacompact space with a G_δ-diagonal such that every relatively discrete subspace has cardinality at most $\mathfrak{c}$, then X is $\mathcal{SM}$-contractible. This enabled them to show that every normal Moore space of cardinality at most $\mathfrak{c}$ is $\mathcal{SM}$-contractible and that every locally connected, locally compact normal Moore space is metrisable.

The following characterisation from [4], a generalisation of Bing's metrisation theorem, was suggested to the author by Reed.

Recall that a *cozero* subset of a space X is the inverse image of $(0, 1]$ under some map $f: X \to [0, 1]$. A collection $\mathscr{C}$ of subsets of X is said to be *separating* if, whenever x and y are distinct points of X,

$$\{C \in \mathscr{C}: x \in C\} \neq \{C \in \mathscr{C}: y \in C\}.$$

For any cardinal λ, the hedgehog of spininess λ, $H(\lambda)$, is the metric space obtained by taking λ copies of the interval $[0, 1]$ and glueing all their left-hand endpoints

[1] The author is indebted to Adam Ostaszewski for suggesting this name.

together (see [5, p. 314]). Points of $H(\lambda)$ are written $[\alpha, r]$, where $\alpha < \lambda$ labels the spine and $r \in [0, 1]$ the distance along it. The zero point on all the spines is the same; it is simply called $[0]$.

Theorem 2.1. *A space X is submetrisable if and only if it has a σ-discrete separating cover of cozero sets.*

Proof. If $(X, \mathcal{T})$ is submetrisable, then its weaker metric topology has a σ-discrete base $\mathcal{B}$. Then $\mathcal{B}$ is σ-discrete with respect to $\mathcal{T}$ and separating, and each member of $\mathcal{B}$ is cozero in the weaker metric topology and so is cozero in $(X, \mathcal{T})$.

Conversely, let $\mathcal{U} = \bigcup \{\mathcal{U}_n : n \in \omega\}$ be a separating cover of cozero sets, where each $\mathcal{U}_n$ is discrete. For each n let $\lambda_n = |\mathcal{U}_n|$ and write $\mathcal{U}_n$ as $\{U_\alpha^n : \alpha < \lambda_n\}$. Let $f_\alpha^n : X \to [0, 1]$ be such that $(f_\alpha^n)^{-1}((0, 1]) = U_\alpha^n$. Let H_n be a hedgehog of spininess λ_n and define $g_n : X \to H_n$ by

$$g_n(x) = \begin{cases} [\alpha, f_\alpha^n(x)], & \text{if } x \in U_\alpha^n, \\ [0], & \text{if } x \notin \bigcup \mathcal{U}_n. \end{cases}$$

This map is well defined and continuous. Let $g : X \to \prod\{H_n : n \in \omega\}$ be the product map

$$g(x) = (g_1(x), g_2(x), g_3(x), \ldots).$$

Then g is continuous, one-to-one and into a metric space, so X is submetrisable. □

Corollary 2.2. (1) *X is $\mathcal{SM}$-contractible if and only if X has a countable separating cover of cozero sets,*

(2) *if X is perfectly normal, then X is submetrisable if and only if X has a σ-discrete separating open cover,*

(3) *if X is perfectly normal, then X is $\mathcal{SM}$-contractible if and only if X has a countable separating open (equivalently closed) cover.*

Note. Corollary 2.2(1) was pointed out by Slaughter, as cited in [9].

Proof. For (1) note that if X has a countable separating cover of cozero sets, then the proof of Theorem 2.1 gives us an injection of X into the Hilbert cube $[0, 1]^\omega$. □

Corollary 2.3 [10]. *If X is perfectly normal, subparacompact, has a G_δ-diagonal and is such that every closed discrete subset has cardinality at most $\mathfrak{c}$, then X is $\mathcal{SM}$-contractible.*

Proof. Let $(\mathcal{G}_n)_{n \in \omega}$ be a G_δ-diagonal sequence for X and let $\{\mathcal{C}_{n,m} : m \in \omega\}$ be a closed refinement of $\mathcal{G}_n$, with each $\mathcal{C}_{n,m}$ discrete. Each $\mathcal{C}_{n,m}$ has cardinality at most $\mathfrak{c}$, so let $f_{n,m} : \mathcal{C}_{n,m} \to 2^\omega$ be an injection. For each finite $\sigma : \omega \to 2$ (σ is *finite* if it has a finite domain) let $C_{n,m,\sigma} = \bigcup f_{n,m}^{-1}([\sigma])$, where $[\sigma] = \{x \in 2^\omega : \sigma \subseteq x\}$. Then

$\{C_{n,m,\sigma}: n, m \in \omega, \sigma: \omega \to 2 \text{ finite}\}$ is a countable closed separating cover so by Corollary 2.2(3) X is $\mathscr{S}\mathscr{M}$-contractible. $\square$

3. Nonstrandable spaces

We define a nonstrandable space to be an uncountable regular Hausdorff space which has no uncountable submetrisable subset. First some elementary

Observations. (1) An uncountable subspace of a nonstrandable space is nonstrandable;

(2) if $(X, \mathscr{T})$ is nonstrandable and if $\mathscr{U}$ is a topology on X weaker than $\mathscr{T}$, then $(X, \mathscr{U})$ is nonstrandable;

(3) if X is a preimage of an uncountable metric space, then X is not a nonstrandable space.

We now give some equivalent definitions of a nonstrandable space.

Theorem 3.1. *X is nonstrandable if and only if the following two conditions hold:*

(1) X is hereditarily ccc,

(2) for any uncountable subset Y of X, if $\mathscr{U}$ is a countable collection of cozero sets in Y, then $\mathscr{U}$ separates just countably many points (where $\mathscr{U}$ separates $S \subseteq Y$ if and only if whenever x and y are two different points of S, $\{U \in \mathscr{U}: x \in U\} \neq \{U \in \mathscr{U}: y \in U\}$).

Proof. If X is not hereditarily ccc, then X has an uncountable subset Y that is not ccc and that therefore has an uncountable collection of pairwise disjoint open sets. Take one point from each to form Z; then Z is an uncountable subset of X and the topology on Z is discrete, so Z is submetrisable. Also if there is an uncountable $Y \subseteq X$ and a countable collection $\mathscr{U}$ of cozero subsets of Y that separate some uncountable $S \subseteq Y$, then S is submetrisable (Theorem 2.1). Conversely suppose that X is not a nonstrandable space. Then there is an uncountable subset Y of X that is submetrisable, i.e., there is a bijection $f: Y \to M$, for some metric space M. If M has a countable base, then the elements of that base are cozero sets in M, so their inverse images under f are cozero in Y, and they separate Y; if not then M contains an uncountable relatively discrete subset S, and then $f^{-1}(S)$ is relatively discrete in X. $\square$

Corollary 3.2. *A sufficient condition for X to be nonstrandable is that no countable collection of open subsets of X separates uncountably many points. This condition is also necessary if X is perfectly normal.*

Proof. Any relatively discrete subset of X of cardinality $\aleph_1$ is $\mathcal{SM}$-contractible and so can be separated by countably many open sets. $\square$

The author is indebted to Robin Knight for some of the following proof.

Theorem 3.3. *An uncountable space X is nonstrandable if and only if every map from an uncountable subset of X to $\mathbb{R}$ has countable image.*

Proof. If there is some uncountable $Y \subseteq X$ and a map $f: Y \to \mathbb{R}$ with uncountable image, then taking one point from every fibre of f we get an uncountable subset Z of Y with an injection to $\mathbb{R}$, so Z is submetrisable and X not nonstrandable. Conversely, suppose that X is not nonstrandable. So there is an uncountable $Y \subseteq X$ and an injection $f: Y \to M$ for some second countable metric space M. Then M can be embedded as a subset of the Hilbert cube I^ω, say $g: M \to I^\omega$ is an injection. If there exists n such that $\pi_n(g(M))$ is uncountable (where π_n denotes the projection of I^ω onto its nth coordinate), then $\pi_n \circ g \circ f$ is a map from Y to $\mathbb{R}$ with uncountable image, violating Observation (3). So assume that $\pi_n(g(M))$ is countable for all $n \in \omega$.

Any countable subset of $\mathbb{R}$ can be embedded in $\mathbb{Q}$, and therefore in $\mathbb{P}$, where $\mathbb{P}$ denotes the irrationals (which are homeomorphic to ω^ω). Taking a product of such injections, we have an embedding of $g(M)$ into $\mathbb{P}^\omega \cong \omega^{\omega \times \omega} \cong \omega^\omega \cong \mathbb{P} \subseteq \mathbb{R}$, and hence a map from Y to $\mathbb{R}$ with uncountable image, again violating Observation (3). So if X is not nonstrandable, then there is a map from a subset of X to $\mathbb{R}$ with uncountable image, establishing the theorem. $\square$

Corollary 3.4. (1) *If X is completely regular and nonstrandable, then X is zero-dimensional.*

(2) *If X is normal and nonstrandable, then X is strongly zero-dimensional.*

Proof. (1) If $x \in U$, where U is open in X, then, since X is completely regular, there is a map $f: X \to [0, 1]$ with $f(x) = 1$ and $f(X - U) = \{0\}$. By Observation (3) f is not onto. But if $r \notin \mathrm{Im}(f)$, then $f^{-1}((r, 1])$ is clopen, contains x and is contained in U, so X has a clopen base. (2) is similar. $\square$

Remark 3.5. If X has cardinality $\aleph_1$ and is zero-dimensional, then there is a one-to-one map f from X to 2^{ω_1}, so there is an uncountable subset of 2^{ω_1} which has a weaker topology than that on X. Thus by Observation (2) if there is a completely regular nonstrandable space, then there is one that is a subset of 2^{ω_1}.

4. Hereditarily ccc and hereditarily Lindelöf spaces

We show here that the search for nonstrandable spaces can be narrowed down to two classes of spaces: hereditarily separable and hereditarily Lindelöf spaces.

Recall that an *S-space* is a space that is hereditarily separable but not Lindelöf and an *L-space* is one that is hereditarily Lindelöf but not separable. Both exist under CH. It is consistent with MA+$\neg$CH that first countable S-spaces exist [1], but MA+$\neg$CH destroys first countable L-spaces [12]. PFA destroys S-spaces [11, p. 323]; it is not known whether "there are no L-spaces" is consistent.

Proposition 4.1. *If X is either hereditarily separable or hereditarily Lindelöf, then X is hereditarily ccc.*

Proof. A relatively discrete uncountable subset is neither separable nor Lindelöf. $\square$

Theorem 4.2. *If X is hereditarily ccc and is not hereditarily separable, then X has an uncountable subspace that is an L-space. If X is hereditarily ccc and is not hereditarily Lindelöf, then X has an uncountable subspace that is an S-space.*

Proof. If X is not hereditarily separable, then X has a left-separated subspace. But any hereditarily ccc left-separated space is an L-space. The second half is similar. $\square$

Corollary 4.3. *If X is a locally countable nonstrandable space, then X has an uncountable subset that is an S-space.*

The above propositions give

Theorem 4.4. *Let X be a nonstrandable space. Then either X has an uncountable subspace Y that is an S-space, or one that is an L-space, or X is hereditarily separable and hereditarily Lindelöf.*

Since it is consistent that there are no S-spaces, it is consistent that every nonstrandable space has an uncountable hereditarily Lindelöf subspace. But in hereditarily Lindelöf spaces the following two well-known propositions hold:

Proposition 4.5. *If X is regular and hereditarily Lindelöf, then X is perfectly normal.*

Proof. Every regular Lindelöf space is normal, so X is hereditarily normal. Let $U \subseteq X$ be open. For each $x \in U$ let V_x be an open set such that $x \in V_x \subseteq \bar{V}_x \subseteq U$. Now U is Lindelöf, so the open cover $\{V_x : x \in U\}$ of U has a countable subcover $\{V_x : x \in S\}$, where $S \subseteq U$ is countable. Then $U = \bigcup \{\bar{V}_x : x \in S\}$, so U is an F_σ set. Thus X is perfect. $\square$

Proposition 4.6. *If X is hereditarily Lindelöf and if $\mathcal{B}$ is any basis for X, then every open subset of X is a countable union of sets from $\mathcal{B}$.*

Proof. If $U \subseteq X$ is open, then U is a union of sets from $\mathcal{B}$. But U is Lindelöf, so there is a countable subcover of U by sets from $\mathcal{B}$. $\square$

Hence, by Corollary 3.2, a hereditarily Lindelöf space X is nonstrandable if and only if no countable collection of members of $\mathcal{B}$ separates uncountably many points (where $\mathscr{B}$ is any basis for X). Now by Remark 3.5, if there is a hereditarily Lindelöf nonstrandable space, then there is one that is a subset of 2^{ω_1}. This gives us the following criterion for the existence of a hereditarily Lindelöf nonstrandable space:

Corollary 4.7. *There is a hereditarily Lindelöf nonstrandable space if and only if there is a hereditarily Lindelöf subset X of 2^{ω_1} such that for each $\alpha \in \omega_1$, $\{x|_\alpha : x \in X\}$ is countable (where $x|_\alpha$ denotes the restriction of x, considered as a function from ω_1 to 2, to the domain α).*

Finally, we make some remarks about product spaces. An infinite product of spaces (each with at least two points) cannot be a nonstrandable space since it contains a copy of the Cantor set. Also a finite product $X_1 \times X_2 \times \cdots \times X_n$ of spaces contains a nonstrandable space only if one of the X_i does. For if $Z \subseteq X_1 \times X_2 \times \cdots \times X_n$ is uncountable, then for some i, $\pi_i(Z)$ is uncountable. If X_i contains no nonstrandable space, then there is an uncountable submetrisable $Y \subseteq \pi_i(Z)$. Then $Z \cap \pi_i^{-1}(Y)$ is a preimage of an uncountable metric space so contains an uncountable submetrisable subset.

5. Examples

The Ostaszewski line L, which exists under $\diamondsuit$, is a perfectly normal S-space that has the property that every open subset of it is either countable or cocountable (see [11]). This implies that it is a nonstrandable space. For by Corollary 3.2 L is nonstrandable if and only if no countable collection of open subsets of L separates uncountably many points of L. Let U_n for $n \in \omega$ by any open subsets of L. Let

$$V_n = \begin{cases} U_n, & \text{if } U_n \text{ is countable,} \\ L - U_n, & \text{otherwise.} \end{cases}$$

Then $\bigcup_{n \in \omega} V_n$ is countable and $\{U_n : n \in \omega\}$ can separate at most one point more than $\bigcup_{n \in \omega} V_n$. So L is a nonstrandable space.

We note that one canonical example of L-spaces cannot give a nonstrandable space: an HFC is a subset of 2^{ω_1} such that whenever $\sigma_1, \sigma_2, \ldots$ are finite functions from ω_1 to 2 with disjoint domains, $X - \bigcup\{[\sigma_n]: n \in \omega\}$ is countable (where $[\sigma_n]$ denotes $\{x \in 2^{\omega_1}: x(\alpha) = \sigma_n(\alpha)$ for all $\alpha \in \operatorname{Dom} \sigma_n\}$). HFCs are L-spaces but not nonstrandable spaces. For let X be an L-space and a nonstrandable space. By Corollary 4.7, there is some countable function $f: \omega_1 \to 2$ such that $\{x \in X: x$ extends $f\}$ is uncountable. For $\alpha \in \operatorname{Dom} f$ let σ_α be the one-place function given by $\sigma_\alpha(\alpha) = 1 - f(\alpha)$. Then there are uncountably many points of X not in $\bigcup\{[\sigma_\alpha]: \alpha \in \operatorname{Dom} f\}$, so X is not an HFC.

Another important example of a nonstrandable space is the Suslin line: Roitman first pointed out to the author that the set of branches b of any Suslin tree T, topologised so that sets $\{b: t \in b\}$ for $t \in T$ form a basis, has a nonstrandable subspace. In the next section we shall put this in context.

6. Nonstrandability of GO spaces

A GO space or *generalised ordered* space (see [8] for details) is a subspace of a linearly ordered space; equivalently, a GO space is a space $(X, \mathcal{T})$ together with an order $<$ on X such that $(X, \mathcal{T})$ includes the linear topology with respect to $<$ and has a base of $<$-convex sets. A GO space is called *Suslin* if it is ccc and nonseparable (that is if it is a dense subspace of a Suslin line). The Suslin hypothesis (SH) is (equivalent to) the assertion that Suslin GO spaces do not exist. Suslin GO spaces exist under $\diamondsuit$ and are destroyed by MA$+\neg$CH.

We make use of the following lemmas.

Lemma 6.1. *Every separable GO space is hereditarily Lindelöf, and therefore is perfectly normal.*

Lemma 6.2. *Every uncountable separable GO space has an uncountable submetrisable subset.*

Proof. Let $(X, \mathcal{T})$ have countable dense subset D. Let $\mathcal{U} = \{(\leftarrow, d): d \in D\}$. By Lemma 6.1 and Corollary 3.2, it is sufficient to show that $\mathcal{U}$ separates uncountably many points. Note that "$\mathcal{U}$ does not separate x and y" is an equivalence relation on X. Suppose that $x<y<z$ are in the same equivalence class. Then $(x, z) \neq \emptyset$ so there exists $d \in (x, z)$. But $(\leftarrow, d)$ separates x and z, a contradiction. So each equivalence class has at most two members, so X can be written as a union of two sets both point-separated by $\mathcal{U}$ and hence both submetrisable. $\square$

The Alexandroff double-arrow space (the space $[0,1] \times \{0,1\}$ with the lexicographic order topology) is an example of a separable linearly ordered space that is not submetrisable but that is a union of two submetrisable subsets, namely the two copies of $[0,1]$.

Lemma 6.2 shows that a nonstrandable GO space must be ccc and nonseparable, and is therefore Suslin. To show conversely that every Suslin line has a nonstrandable subspace, we make use of the following theorem of Przymusinski.

Theorem 6.3 [2]. *If X is a submetrisable GO space under the ordering $<$, then X has a metric subtopology that is also a GO space under the ordering $<$.*

Lemma 6.4. *Every ccc submetrisable GO space is separable.*

Proof. Let $(X, \mathcal{T})$ be a ccc GO space, and let $\mathcal{M}$ be a metric GO subtopology with the same underlying order $<$. $\mathcal{M}$ is also ccc and is therefore separable. Let D be a countable dense subset and let I be the set of isolated points of $\mathcal{T}$. As $(X, \mathcal{T})$ is ccc, I is countable, and we can show that $D \cup I$ is dense in $(X, \mathcal{T})$. For if the convex set $T \in \mathcal{T}$ has at least three points, say $a<b<c$, then (a, c) is a nonempty set that

is open in the linear topology and so in $\mathcal{M}$, so (a, c) meets D. As T is convex, T meets D. Also if $T \in \mathcal{T}$ is nonempty and has less than three points, then T contains an isolated point and so meets I. So each nonempty convex set in $\mathcal{T}$ meets $D \cup I$ and so, as $\mathcal{T}$ has a basis of convex sets, $\mathcal{T}$ is separable. □

Thus we have the following theorem.

Theorem 6.5. *A GO space is nonstrandable if and only if it is Suslin and has no uncountable separable subspaces. Every Suslin GO space has a nonstrandable subspace.*

Proof. We have already shown that if a GO space is nonstrandable, then it is Suslin, and by Lemma 6.2 it can have no uncountable separable subsets. Conversely if a GO space X is ccc, then it is hereditarily ccc. For suppose that D is a relatively discrete subset of X, so for each $d \in D$ there is an interval (x_d, y_d) such that $D \cap (x_d, y_d) = \{d\}$. The intervals (d, y_d) for $d \in D$ are disjoint so, since X is ccc, only countably many of them are nonempty. Similarly only countably many of the intervals (x_d, d) are nonempty. So all but countably many $d \in D$ are isolated, hence D is countable as X is ccc. So a Suslin GO space with no uncountable separable subsets is hereditarily ccc and therefore, by Theorem 4.2, it is an L-space. By Lemma 6.4, a GO L-space has no uncountable submetrisable subspaces. Finally every Suslin GO space is hereditarily ccc and nonseparable, and so must have a left-separated subspace. This can have no uncountable separable subspaces. □

This leads to the following alternative characterisation of nonstrandability in the presence of the Suslin hypothesis.

Theorem 6.6 (SH). *A space X is nonstrandable if and only if it has no uncountable subspaces with a sublinear topology (i.e., with a topology that includes the order topology with respect to some linear order).*

Proof. Theorem 6.5 shows that under SH an uncountable subspace with a sublinear topology will have an uncountable submetrisable subset, while Theorem 3.3 shows that if X is not nonstrandable, then it will have a subspace that is an injective preimage of an uncountable subset of $\mathbb{R}$, and whose topology must therefore be sublinear. □

7. Some spaces that are not nonstrandable spaces

We give here proofs that various classes of spaces are not nonstrandable spaces. Any normal Čech-complete space that is nonstrandable must contain an S-space, as must any space with a G_δ-diagonal. So under PFA, for example, no nonstrandable space is Čech-complete or has a G_δ-diagonal. Also a perfectly normal space cannot be nonstrandable if it has a G_δ^*-diagonal or if it has a G_δ-diagonal and is second category.

Theorem 7.1. *If X is a normal nonstrandable Čech-complete space, then X has an uncountable subset Y that is locally countable, and therefore an S-space.*

Proof. Let $(\mathcal{G}_n)_{n\in\omega}$ be a sequence of open covers of X witnessing Čech-completeness; thus, whenever $(M_n)_{n\in\omega}$ is a sequence of closed sets such that for each n there exists $G_n \in \mathcal{G}_n$ with $M_n \subseteq G_n$, the intersection $\bigcap_{n\in\omega} M_n$ is nonempty. Suppose that no uncountable subset of X is locally countable (otherwise, by Corollary 4.3, X has an S-subspace). Let $Y = X - \bigcup\{U\colon U$ is open in X and countable$\}$. So Y is X with just countably many points removed, and every open set in X that meets Y is uncountable. Note that by regularity and the Hausdorff property, given any open $U \subseteq X$ meeting Y we can find nonempty open V and W meeting Y with disjoint closures such that $\bar{V} \cup \bar{W} \subseteq U$.

Let $I_n = \{f\colon n \to \{0, 1\}\}$, and let $I = \bigcup_{n\in\omega} I_n$. We define nonempty open sets U_f, for $f \in I$, as follows. Let $U_\emptyset = X$. Suppose that U_f is defined and meets Y for $f \in I_m$ for all $m < n$. Take $f \in I_n$ and let $g = f|_{n-1}$. Let $f'\colon n \to \{0, 1\}$ be such that $f'|_{n-1} = g$ and $f' \neq f$ (so that f and f' are the only functions in I_n that extend g). By the above, we may pick nonempty open sets U_f, $U_{f'}$ with disjoint closures such that $\overline{U_f} \cup \overline{U_{f'}} \subseteq U_g$, and such that U_f, $U_{f'}$ both meet Y and both refine elements of $\mathcal{G}_n$.

Now let $J = \{f\colon \omega \to \{0, 1\}\}$. For each $f \in J$, $\bigcap_{n\in\omega} U_{f|n} \supseteq \bigcap_{n\in\omega} \overline{U_{f|n+1}} \neq \emptyset$, by Čech-completeness, so pick $x_f \in \bigcap_{n\in\omega} U_{f|n}$.

Then the points x_f for $f \in J$ are all distinct and, moreover, are separated by $\{U_g : g \in I\}$, which is countable. Those sets might not be cozero. But for each $g \in I$ let g_1 and g_2 be the two one-place extensions of g; then $\overline{U_{g_1}} \cup \overline{U_{g_2}} \subseteq U_g$ so by normality there exists $h_g : X \to [0, 1]$ such that $h_g(\overline{U_{g_1}} \cup \overline{U_{g_2}}) = \{1\}$ and $h_g(X - U_g) = \{0\}$. Then the countable collection of functions $\{h_g : g \in I\}$ separates $(x_f : g \in J\}$ and so $\{x_f : f \in J\}$ is $\mathcal{SM}$-contractible. □

A space X is said to have a *G_δ-diagonal* if there is a sequence $(\mathcal{G}_n)_{n\in\omega}$ of open covers of X such that, whenever x and y are distinct points of X, there exists n such that $y \notin \mathrm{St}(x, \mathcal{G}_n)$ (where $\mathrm{St}(x, \mathcal{G}_n) = \bigcup\{G \in \mathcal{G}_n : x \in G\}$).

Proposition 7.2. *If X has a G_δ-diagonal and is nonstrandable, then X is an S-space.*

Proof. If X is nonstrandable but not an S-space, then by Theorem 4.2, X has an uncountable Lindelöf subset Y. But then, since X is regular, Y is paracompact and so submetrisable. □

Theorem 7.3. *If X is perfectly normal and has a G_δ^*-diagonal, then X is not a nonstrandable space.*

Proof. X has a G_δ^*-diagonal if it has a G_δ-diagonal sequence $(\mathcal{G}_n)_{n\in\omega}$ such that

$$x = \bigcap_{n\in\omega} \overline{\mathrm{St}(x, \mathcal{G}_n)}.$$

By Proposition 7.2, we may assume that X is separable. Let S be a countable dense subset in X. Let $\mathcal{U}_n = \{\mathrm{St}(s, \mathcal{G}_n) : s \in S\}$, and let $\mathcal{U} = \bigcup \{\mathcal{U}_n : n \in \omega\}$. $\mathcal{U}$ is a countable collection of open subsets of X, and each $\mathcal{U}_n$ covers X. Suppose that x and y are distinct points of X, and suppose that x and y are not separated by $\mathcal{U}$. Then, for each $s \in S$ and $n \in \omega$,

$$x \in \mathrm{St}(s, \mathcal{G}_n) \quad \text{if and only if} \quad y \in \mathrm{St}(s, \mathcal{G}_n),$$

i.e.,

$$s \in \mathrm{St}(x, \mathcal{G}_n) \quad \text{if and only if} \quad s \in \mathrm{St}(y, \mathcal{G}_n)$$

so

$$\mathrm{St}(x, \mathcal{G}_n) \cap S = \mathrm{St}(y, \mathcal{G}_n) \cap S \quad \text{for all } n.$$

So

$$\overline{\mathrm{St}(x, \mathcal{G}_n) \cap S} = \overline{\mathrm{St}(y, \mathcal{G}_n) \cap S},$$

$$\overline{\mathrm{St}(x, \mathcal{G}_n)} = \overline{\mathrm{St}(y, \mathcal{G}_n)},$$

because $\bar{U} = \overline{U \cap S}$ whenever U is open and S dense. Therefore

$$\bigcap_{n \in \omega} \overline{\mathrm{St}(x, \mathcal{G}_n)} = \bigcap_{n \in \omega} \overline{\mathrm{St}(x, \mathcal{G}_n)}$$

so that $x = y$. So $\mathcal{U}$ point-separates X and so X is submetrisable. □

Theorem 7.4. *If X is perfectly normal, second category and has a G_δ-diagonal, then X is not a nonstrandable space.*

Proof. Let $\mathcal{U}_n$, $\mathcal{U}$ be as in the proof above. For $x \in X$ let $[x]$ denote the sets of points of X which are not separated from x by $\mathcal{U}$ (note that "$\mathcal{U}$ does not separate x and y" is an equivalence relation on X). We show that there are uncountably many equivalence classes $[x]$; then picking one point from each we get an uncountable subset that is separated by $\mathcal{U}$ and that therefore is submetrisable.

By the proof of Theorem 7.3, $y \in [x]$ implies

$$\bigcap_{n \in \omega} \overline{\mathrm{St}(x, \mathcal{G}_n)} = \bigcap_{n \in \omega} \overline{\mathrm{St}(y, \mathcal{G}_n)}$$

so in particular $[x] \subseteq \bigcap_{n \in \omega} \overline{\mathrm{St}(x, \mathcal{G}_n)}$.

Now, for each x, $\bigcap_{n \in \omega} \overline{\mathrm{St}(x, \mathcal{G}_n)} = (\bigcap_{n \in \omega} \mathrm{St}(x, \mathcal{G}_n)) \cup (\bigcup_{n \in \omega} \delta(\mathrm{St}(x, \mathcal{G}_n)))$, where δA denotes the boundary of A, so that, for each x, $[x]$ is covered by countably many nowhere dense sets. As X is not a union of countably many nowhere dense sets, there must be uncountably many equivalence classes, so X has an uncountable submetrisable subset. □

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